

Existence of Nash equilibria in sporting contests with capacity constraints

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This article considers a contest model of an n -team professional sports league. The market areas in which teams are located may differ from one another and each team may have different preferences for winning. In a general asymmetric sporting contest, we demonstrate that under standard assumptions, there exists a unique non-trivial Nash equilibrium in which at least two teams must be active in equilibrium. In addition, we prove that at the non-trivial equilibrium, each team's winning percentage and playing talent are determined by its composite strength—market size and win preference.

JEL classification: L83, C72, L13

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1. Introduction

The main purpose of this paper is to demonstrate the existence of pure-strategy Nash equilibria in an “ n ” team sporting contest. Since the seminal papers of Szymanski (2003, 2004) and Szymanski and Késenne (2004), the Nash equilibrium model has been used in the analysis of professional team sports. However, many papers have been restricted to a two-team league model (Chang and Sanders, 2009; Cyrenne, 2009; Dietl *et al.*, 2009). Dietl *et al.* (2008) that is considered a more general n -team league model; however, it is

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based on the assumption that all teams have identical revenue generating potential and cost functions. Thus the sporting contest is symmetric. These restrictions most probably apply to the Nash equilibrium model in sports because of the difficulty in managing non-identical teams with respect to their market size or drawing potential by conventional means, which treat the Nash equilibrium as a fixed point of the best response mapping. This entails working in a dimension space equal to the number of teams. In the present study, we adopt the share function approach introduced in Cornes and Hartley (2005) which extends the inclusive reaction function used by Szidarovszky and Yakowitz (1977) to study Cournot oligopoly games. The advantages of the approach are twofold: one is to avoid the proliferation of dimensions associated with the best response function approach and the other is to be able to analyze sporting contests involving many heterogeneous teams. Following the same steps as in Cornes and Hartley and Hirai and Szidarovszky (2013), we will prove that there exists a unique non-trivial Nash equilibrium in which at least two teams must be active in equilibrium. The uniqueness of Nash equilibrium is an important issue for the non-cooperative game. If the equilibrium is unique, then we have a self-constrained theory for predicting the game's outcome. Moreover, uniqueness is crucial for comparative statics analysis which allows one to obtain qualitative results.

In addition, this study demonstrates that at the non-trivial equilibrium, each team's winning percentage and playing talent are determined by its composite strength, its market size and win preference. The findings' implications are significant for the premise of competitive-balance rules such as revenue sharing and salary caps. It has been recognized that unrestricted competition between teams will lead to a league dominated by a few large-market teams with strong-drawing potential. In the theoretical literature on sports contests, however, this situation is not self-evident. Szymanski and Késenne (2004, p. 169) demonstrated that if there is no revenue sharing in equilibrium, a large-market team will dominate a small one in a two-team league. However, Késenne (2005, p. 103) observed that this result does not necessarily hold in an n -team model. Moreover, Késenne (2007, pp. 54-55) and Dietl *et al.* (2011) demonstrated that if team objectives

maximize a combination of profits and wins, as introduced by Rascher (1997), a large-market team will not always dominate a small one in equilibrium, but these studies are restricted to two-team models. The contribution of the present study is in unifying and clarifying the results of these studies by putting them into a more general n -team model.

The rest of the paper is organized as follows. Section 2 explains the basic model and the assumptions. In Section 3, we establish the existence of Nash equilibria in an n -team sporting contest. In this section, we also compare the winning percentage and playing talent of teams of different market sizes and win preferences. Concluding remarks are presented in Section 4.

2. The Model

We consider a professional sports league consisting of $n(\geq 2)$ teams where each team $i(= 1, \dots, n)$ independently chooses a level of talent, $t_i(\geq 0)$, to maximize the objective function. By assuming a competitive labor market and following the sports economics literature, talent can be hired in the players' labor market at a constant marginal cost $c > 0$; hence, the cost function can be written as

$$C_i(t_i) = ct_i. \quad (1)$$

On the revenue side, the season revenue function of a team is defined as

$$R_i = R_i(w_i). \quad (2)$$

R_i is total season revenue of team i , w_i is the winning percentage of the team. It is common in the sports economics literature to assume the following.

Assumption 1. For all i , the function R_i satisfies $R_i(0) = 0$ and $R_i(w_i) > 0$ for $w_i \in (0,1]$. Moreover, R_i is twice differentiable and either satisfies $R_i' > 0$ and $R_i'' \leq 0$ for all $w_i \in [0,1]$, or there exists a $\bar{w}_i \in (1/n, 1]$ such that if $w_i \geq \bar{w}_i$, then $R_i' < 0$; otherwise, $R_i' > 0$, and $R_i'' < 0$ elsewhere.

This assumption reflects the uncertainty of outcome hypothesis (Rottenberg, 1956; Neale, 1964) that consumers in aggregate prefer a close match to one that is unbalanced in favor of one of the teams.

The win percentage is characterized by the contest success function (CSF). The most widely used functional form in sporting contests is the logit that can be written as

$$w_i(t_1, \dots, t_n) = \begin{cases} \frac{n}{2} \frac{t_i}{t_i + T_{-i}} & \text{if } t_i > 0 \text{ and } T_{-i} > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where $T_{-i} = \sum_{j \neq i}^n t_j$.¹ The factor $n/2$ results from the fact that winning percentages must average to $1/2$ within a league during any one year; that is, $\frac{1}{n} \sum_{i=1}^n w_i = 1/2$. Notice that for the two-team models, the logit CSF (3) does not place a restraint on the teams' choices. However, for the n -team models this is not the case with the logit CSF (3). More precisely, the winning percentage can be larger than one if a team holds more than $2/n$ per cent of total league talent (with normalization of $\sum_{i=1}^n t_j$ to one). So, as in Szymanski (2010), we introduce an extra constraint:

$$w_i(t_1, \dots, t_n) = \min\left\{\frac{n}{2} \frac{t_i}{t_i + T_{-i}}, 1\right\}, \quad (4)$$

1 The logit CSF was explicitly adopted in the seminal work of El-Hodiri and Quirk (1971). Groot (2008, pp. 97-100) has expressed the season winning percentage as follows: $w_i = \frac{t_i}{n-1} (\sum_{j \neq i}^n \frac{1}{t_i + t_j})$. Although this equation gives the correct relationship between winning percentage and team quality, it considerably complicates the derivative of the marginal product of talent. We therefore choose the simple approximation of the winning percentage (3).

that is, when the number of teams exceeds by two, if $t_i > \frac{2T-i}{n-2}$, then the win percentage of team i is always one (see Figure 1).

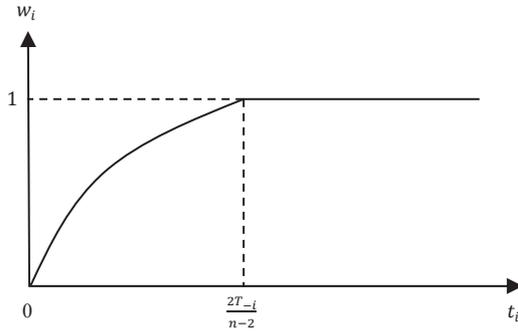


Figure 1 Shape of the winning percentage of team i

Then, the profit of team i is described by

$$\pi_i(t_1, \dots, t_n) = R_i(w_i) - ct_i. \quad (5)$$

As in Rascher (1997), Késenne (2007, p.5), and Dietl *et al.* (2011), the objective function of team i is given by a linear combination of profits and wins, which can be written as

$$u_i(t_1, \dots, t_n) = \pi_i + \gamma_i w_i = \begin{cases} R_i(w_i) - ct_i + \gamma_i w_i & \text{if } 0 \leq t_i \leq \frac{2T-i}{n-2}, \\ R_i(1) - ct_i + \gamma_i & \text{if } t_i > \frac{2T-i}{n-2}, \end{cases} \quad (6)$$

where $\gamma_i \geq 0$ is the weight parameter that characterizes the weight team i places on winning in the objective function.² The objective function allows teams to be more profit

2 Note that Sloane (1971) was the first to suggest that the owner of a sports team actually maximizes utility, which may include inter alia playing success and profits. Garcia-del-Barrio and Szymanski (2009) empirically found that the behavior of clubs in English and Spanish leagues over the period 1994-2004 seem to closely approximate win maximization subject to a financial constraint.

oriented or more win oriented because the weight parameter γ_i can be different for every team. We refer to this objective function as the payoff function of team i . The objective of each team is to maximize u_i with respect to t_i subject to $t_i \in [0, L_i]$, where L_i is the *capacity limit* of team i due to financial constraints.³ Our analysis of the sports league is formulated as a simultaneous-move game and the solution concept we use throughout the study is that of a pure-strategy Nash equilibrium.

Finally, it is occasionally assumed that the total supply of talent is fixed in the analysis of sports leagues. Researchers who have made this assumption have used a non-Nash conjecture to reflect this scarcity in each team's first-order condition (Fort and Quirk, 1995; Vrooman, 1995). In this case and for a two-team league, we have $\frac{dt_2}{dt_1} = -1$. Indeed, although opinion is divided among sports economists on this subject, we use the Nash conjecture $\frac{dt_2}{dt_1} = 0$ in this study (see e.g., Eckard, 2006; Szymanski, 2004, 2006; Vrooman, 2007). The main reason is that as Szymanski (2004) acutely pointed out, as far as modeling the game is concerned, there is no inconsistency between the use of Nash conjecture and the assumption that supply of talent is fixed.

3. Equilibrium Analysis

We can now calculate the best response of team i . Assume first that $T_{-i} = 0$ in order that the other teams do not spend any resources on playing talent. Then, if $t_i > 0$, the payoff is negative in light of Assumption 1 and CSF (3). If team i sets $t_i = 0$, the payoff becomes zero. Therefore, this game always has a trivial equilibrium point $\bar{t}_1 = \bar{t}_2 = \dots = \bar{t}_n = 0$. Our concern is with non-trivial equilibria (i.e., $\sum_{i=1}^n \bar{t}_i > 0$) and thus no further consideration is given to the trivial point.

If $T_{-i} > 0$, it follows from payoff function (6) that we have

³ Hirai and Szidarovszky (2013) analyze asymmetric contests with budget-constrained players.

$$\frac{\partial}{\partial t_i} u_i(t_1, \dots, t_n) = \begin{cases} (R'_i(w_i) + \gamma_i) \frac{n}{2} \frac{T_{-i}}{(t_i + T_{-i})^2} - c & \text{if } 0 \leq t_i < \frac{2T_{-i}}{n-2}, \\ -c & \text{if } t_i > \frac{2T_{-i}}{n-2}, \end{cases} \quad (7)$$

and this derivative does not exist if $t_i = 2T_{-i}/(n-2)$ for $n > 2$. As the second-order condition we obtain

$$\frac{\partial^2}{\partial t_i^2} u_i(t_1, \dots, t_n) = \frac{nT_{-i}}{(t_i + T_{-i})^3} (R''_i(w_i)) \frac{nT_{-i}}{4(t_i + T_{-i})} - (R'_i(w_i) + \gamma_i) < 0. \quad (8)$$

Under Assumption 1, the second-order condition (8) is satisfied. Therefore, team i 's best response function $t_i = \phi_i(T_{-i})$ is well defined. Notice first that the best response of this team cannot exceed $2T_{-i}/(n-2)$ for $n > 2$. In contrast, assume that $t_i > 2T_{-i}/(n-2)$, then its payoff is given by the second case of (6). By decreasing the value of t_i by a small amount, its revenue stays the same, the parameter γ_i is same and cost decreases. So the payoff of this team would increase contradicting the assumption that $t_i = \phi_i(T_{-i})$ is the team i 's best response. Therefore with fixed values of $T_{-i} > 0$ the best response of team i is selected in the $[0, \tilde{L}_i]$ with $\tilde{L}_i = \min\left\{L_i, \frac{2T_{-i}}{n-2}\right\}$. If the capacity limits of the teams are sufficient small, that is, $L_i < \frac{2T_{-i}}{n-2}$ for all i and T_{-i} , then the constant segment of the winning percentage (4) cannot occur for all teams. For the sake of simplicity in the following discussion we will assume that this is the case. Hence, it follows from equation (7) that given $T_{-i} > 0$, team i 's best response function $t_i = \phi_i(T_{-i})$ is given by

$$\phi_i(T_{-i}) = \begin{cases} 0 & \text{if } \left. \frac{\partial u_i}{\partial t_i} \right|_{t_i=0} = (R'_i(0) + \gamma_i) \frac{n}{2T_{-i}} - c \leq 0, \\ L_i \left(< \frac{2T_{-i}}{n-2} \right) & \text{if } \left. \frac{\partial u_i}{\partial t_i} \right|_{t_i=L_i} = (R'_i(w_i) + \gamma_i) \frac{T_{-i}}{2(L_i + T_{-i})^2} - c \geq 0, \\ t_i^* & \text{otherwise,} \end{cases} \quad (9)$$

where t_i^* is the unique solution of the strictly monotonic equation

$$\frac{\partial u_i}{\partial t_i} = (R'_i(w_i) + \gamma_i) \frac{n}{2} \frac{T_{-i}}{(t_i + T_{-i})^2} - c = 0 \quad (10)$$

in interval $(0, L_i)$ (see Figure 2).

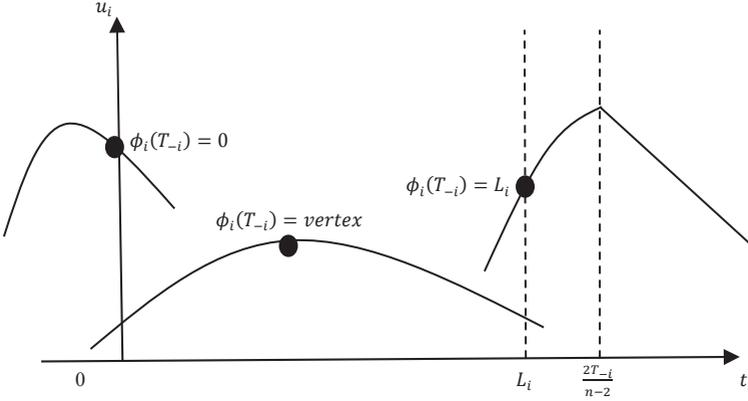


Figure 2 Shape of the payoff function of team i

Observe that because of Assumption 1, the left-hand side of equation (10) strictly decreases and is continuous in t_i , positive at $t_i = 0$ and negative at $t_i = L_i$, therefore there is a unique solution, $\phi_i(T_{-i})$. It is well known that a strategy profile $(\bar{t}_1, \dots, \bar{t}_n)$ is an equilibrium if and only if for all i , \bar{t}_i is the best response with fixed values of \bar{T}_{-i} , that is,

$$\bar{t}_i = \phi_i(\bar{T}_{-i}) \quad \text{with} \quad \bar{T}_{-i} = \sum_{j \neq i}^n \bar{t}_j \quad \text{for all } i. \quad (11)$$

However, application of the conventional best response function approach entails working in a dimension space equal to the number of teams. This significantly complicates the analysis of sporting contests with many heterogeneous teams. We then can rewrite the best responses of the teams in terms of aggregate talent, which we will

denote by $T = \sum_{i=1}^n t_i$. That is,

$$\Phi_i(T) = \phi_i(T - \Phi_i(T)). \tag{12}$$

Following Wolfstetter (1999, p. 91), we call $\Phi_i(T)$ the *inclusive reaction function* of team i , which is proposed by Szidarovszky and Yakowitz (1977).⁴ From equation (9), we have

$$\Phi_i(T) = \begin{cases} 0 & \text{if } (R'_i(0) + \gamma_i) \frac{n}{2T} - c \leq 0, \\ L_i & \text{if } (R'_i(w_i) + \gamma_i) \frac{(T - L_i)}{2T^2} - c \geq 0, \\ t_i^{**} & \text{otherwise,} \end{cases} \tag{13}$$

where t_i^{**} solves equation

$$\frac{n}{2}(R'_i(w_i) + \gamma_i) \left(1 - \frac{t_i}{T}\right) - cT = 0 \tag{14}$$

in interval $(0, L_i)$. Notice that in the third case of (13), the left-hand side of equation (14) is positive at $t_i = 0$, negative at $t_i = L_i$, and strictly decreasing, because it has a negative derivative given by

$$\frac{\partial}{\partial t_i} \left\{ \frac{n}{2}(R'_i(w_i) + \gamma_i) \left(1 - \frac{t_i}{T}\right) - cT \right\} = \frac{n^2 R''_i(w_i)}{4T} \left(1 - \frac{t_i}{T}\right) - \frac{n}{2T} (R'_i(w_i) + \gamma_i) < 0,$$

where the sign comes from Assumption 1. Therefore, there is a unique solution of equation (14), which is a continuously differentiable function of $T > 0$ by the implicit

4 Cornes and Hartley (2005) called this function *replacement function*.

function theorem.

In view of (11) and (12), it is easy to see that Nash equilibrium values of T are the solution of the equation

$$\sum_{i=1}^n \Phi_i(T) - T = 0. \tag{15}$$

The left-hand side, denoted by $H(T)$, has the following properties. It is continuous, since all $\Phi_i(T)$ are continuous, $H(T) \geq 0$ for sufficiently small values of T , since $\Phi_i(T) \geq 0$ for all i and, $H(\sum_{i=1}^n L_i) \leq 0$ since $\Phi_i(T) \leq L_i$. Therefore, there is at least one solution. Furthermore, if Φ_i is strictly monotone decreasing in interval $(0, L_i)$, then equation (15) has a unique solution \bar{T} , and the corresponding equilibrium talents are $\bar{t}_i = \Phi_i(\bar{T})$.

So, implicitly differentiating equation (14) with respect to T and considering $t_i = \Phi_i(T)$, we have

$$\frac{n^2}{4} R_i'' \frac{\Phi_i' T - t_i}{T^2} \left(1 - \frac{t_i}{T}\right) - \frac{n}{2} (R_i' + \gamma_i) \frac{\Phi_i' T - t_i}{T^2} - c = 0,$$

implying that

$$\Phi_i'(T) = \frac{4cT^2 + nt_i(R_i''(1 - t_i/T) - 2(R_i' + \gamma_i))}{nT(R_i''(1 - t_i/T) - 2(R_i' + \gamma_i))}.$$

Here the denominator is negative but the sign of the numerator is not determined by Assumption 1. Hence, $\Phi_i'(T)$ is not necessarily monotonic. This fact creates slight additional difficulties in order to prove uniqueness of non-trivial Nash equilibrium.⁵

5 Notice that the best response function $\phi_i(T_{-i})$ is not necessarily monotonic either, which stands in contrast to monotonic best response functions: either decreasing---i.e., strategic substitution---or increasing ---i.e., strategic complementarity (see Bulow *et al.*, 1985).

So, we will examine the properties of team i 's *share function* $S_i(T) = \Phi_i(T)/T$, proposed by Cornes and Hartley (2005). As stated above, the inclusive reaction function differs from the best response function in that it uses the aggregate talent as the independent variable. The share function shares this feature, but works with a different dependent variable, that is, the team's share of aggregate talent. We define team i 's share value as $s_i = t_i/T$. It follows from team i 's inclusive reaction function (13) that we have

$$S_i(T) = \begin{cases} 0 & \text{if } \frac{n}{2}(R'_i(0) + \gamma_i) - cT \leq 0, \\ \frac{L_i}{T} & \text{if } \frac{n}{2}\left(R'_i\left(\frac{nL_i}{2T}\right) + \gamma_i\right)\left(1 - \frac{L_i}{T}\right) - cT \geq 0, \\ s_i^* & \text{otherwise,} \end{cases} \quad (16)$$

where s_i^* is the unique solution of equation

$$\frac{n}{2}\left(R'_i\left(\frac{n}{2}s_i\right) + \gamma_i\right)(1 - s_i) - cT = 0, \quad (17)$$

in interval $(0, L_i/T)$. The left-hand side of equation (17) is positive at $s_i = 0$, negative at $s_i = L_i$, and strictly decreasing, because it has a negative derivative given by

$$\frac{\partial}{\partial s_i} \left\{ \frac{n}{2}\left(R'_i\left(\frac{n}{2}s_i\right) + \gamma_i\right)(1 - s_i) - cT \right\} = \frac{n^2}{4}R''_i(1 - s_i) - \frac{n}{2}\left(R'_i\left(\frac{n}{2}s_i\right) + \gamma_i\right) < 0,$$

where the sign comes from Assumption 1. Therefore, there is a unique solution s_i^* which is differentiable by the implicit function theorem. In our further analysis we will need the derivative of the share function. By differentiating equation (17) with respect to T and considering $s_i = S_i(T)$, we have

$$\frac{n}{2} \left(R_i'' \frac{n}{2} S_i'(1 - s_i) - (R_i' + \gamma_i) S_i' \right) - c = 0,$$

implying that

$$S_i'(T) = \frac{c}{\frac{n^2}{4} R_i''(1 - s_i) - \frac{n}{2} (R_i' + \gamma_i)} < 0. \quad (18)$$

The inequality follows since the numerator is positive and the denominator is negative by Assumption 1. So, $S_i(T)$ is continuous with constant and strictly decreasing segments.

Then, equation (15) can be also rewritten as

$$\sum_{i=1}^n S_i(T) - 1 = 0, \quad (19)$$

where the left-hand side is strictly decreasing in T unless all $S_i(T) = 0$. Suppose that equation (19) can have two different solutions $\bar{T}' > \bar{T} > 0$. Then, at least two teams must be active in the non-trivial Nash equilibrium. Assume therefore that $S_i(\bar{T}') > 0$ ($i = 1, 2$). In this case $S_i(\bar{T}) > S_i(\bar{T}')$ and for all $j \neq i, S_j(\bar{T}) \geq S_j(\bar{T}')$ in light of expression (18). Then

$$\sum_{i=1}^n S_i(\bar{T}) > \sum_{i=1}^n S_i(\bar{T}')$$

which is an obvious contradiction. Therefore, the equilibrium value of $\bar{T} (> 0)$ is unique. Given an equilibrium \bar{T} , the corresponding unique strategy profile $(\bar{t}_1, \dots, \bar{t}_n)$ is found by multiplying \bar{T} by each team's share evaluated at \bar{T} : $\bar{t}_i = \bar{T} S_i(\bar{T})$. Hence we prove that the following result:

Proposition 1. *Under Assumption 1, the sporting contest has a unique non-trivial Nash equilibrium in pure strategies.*

The share function approach is very helpful in proving the existence and uniqueness of the equilibrium, as stated above.⁶ In addition, it provides a simple computational method to find the equilibrium.

Example 1. Assume linear revenue functions, $R_i(w_i) = m_i w_i$ with $m_i > 0$.⁷ The parameter m_i represents the market size of team i . The payoff of team i now has the form

$$u_i(t_1, \dots, t_n) = \begin{cases} (m_i + \gamma_i)w_i - ct_i & \text{if } 0 \leq t_i \leq \frac{2T_{-i}}{n-2}, \\ (m_i + \gamma_i) - ct_i & \text{if } t_i > \frac{2T_{-i}}{n-2}. \end{cases}$$

In this case u_i is strictly concave in t_i with derivative

$$\frac{\partial u_i}{\partial t_i} = \begin{cases} (m_i + \gamma_i) \frac{n}{2} \frac{T_{-i}}{(t_i + T_{-i})^2} - c & \text{if } 0 \leq t_i < \frac{2T_{-i}}{n-2}, \\ -c & \text{if } t_i > \frac{2T_{-i}}{n-2}, \end{cases}$$

and this derivative does not exist if $t_i = \frac{2T_{-i}}{n-2}$ for $n > 2$. For the sake of simplicity we assume again that $L_i < \frac{2T_{-i}}{n-2}$ for all i and T_{-i} , that is, the constant segment of the winning percentage (4) cannot occur for all teams. Since u_i is strictly concave in t_i , the best

6 In the oligopoly literature there is a large set of sufficient conditions for existence and uniqueness, some of which are more and some less technically demanding (see, e.g., Friedman, 1986). However, the share function approach has the advantage that it is accessible without advanced mathematical prerequisites.

7 Szymanski (2010) assumed a quadratic revenue function and presented a simulation model of an n -team sports league.

response of team i is unique and is given as

$$\phi_i(T_{-i}) = \begin{cases} 0 & \text{if } \frac{nv_i}{2T_{-i}} - c \leq 0, \\ L_i & \text{if } \frac{nv_i T_{-i}}{2(L_i + T_{-i})^2} - c \geq 0, \\ \sqrt{\frac{nv_i T_{-i}}{2c}} - T_{-i} & \text{otherwise,} \end{cases} \quad (20)$$

where $v_i = m_i + \gamma_i$. Then, the inclusive reaction function of team i is easily obtained from the best response function (20):

$$\Phi_i(T) = \begin{cases} 0 & \text{if } \frac{nv_i}{2T} - c \leq 0, \\ L_i & \text{if } \frac{nv_i(T - L_i)}{2T^2} - c \geq 0, \\ T - \frac{2cT^2}{nv_i} & \text{otherwise.} \end{cases} \quad (21)$$

By dividing both side of (21) by $T > 0$, the share function of team i is given as

$$S_i(T) = \begin{cases} 0 & \text{if } \frac{nv_i}{2} - cT \leq 0, \\ \frac{L_i}{T} & \text{if } \frac{nv_i}{2} \left(1 - \frac{L_i}{T}\right) - cT \geq 0 \\ 1 - \frac{2cT}{nv_i} & \text{otherwise.} \end{cases} \quad (22)$$

Notice that the three functions are no more than alternative ways of presenting precisely the same information. However, the simple piecewise-linear form of the share function (22) suggests that this is the most convenient to use. It follows from equation (19) that Nash equilibrium values of T occur where the aggregate share function equals unity. By assuming an interior optimum, then from (22),

$$\sum_{i=1}^n S_i(T) = \sum_{i=1}^n \left(1 - \frac{2cT}{nv_i}\right) = 1.$$

So the aggregate talent is

$$\bar{T} = \frac{n(n-1)}{2c \sum_{j=1}^n (1/v_j)},$$

and by substituting it into the third case of (22), the share values of team i are given by

$$\bar{s}_i = S_i(\bar{T}) = 1 - \frac{n-1}{v_i \sum_{j=1}^n (1/v_j)}.$$

Given \bar{T} , the corresponding equilibrium strategy profile is found by multiplying \bar{T} by each team's share evaluated at \bar{T} :

$$\bar{t}_i = \bar{T} S_i(\bar{T}) = \frac{n(n-1)}{2c \sum_{j=1}^n (1/v_j)} \left(1 - \frac{n-1}{v_i \sum_{j=1}^n (1/v_j)}\right).$$

Finally, a team's winning percentage in (3) is determined by the ration of its talent to all the talent in the league. Therefore, the equilibrium win percentage of team i is given by

$$\bar{w}_i = \frac{n}{2} \bar{s}_i = \frac{n}{2} \left(1 - \frac{n-1}{v_i \sum_{j=1}^n (1/v_j)}\right).$$

The approach to share function used in this study is also useful for deriving certain properties of the equilibrium win percentage and playing talent of team i . For simplicity, consider an interior equilibrium in what follows. Then we can establish the following results.

Proposition 2. Suppose Assumption 1 holds for all teams. Then at the non-trivial Nash equilibrium, we have

$$\bar{w}_i [\bar{\ell}_i] \succcurlyeq \bar{w}_j [\bar{\ell}_j] \text{ if and only if } R'_i(\bar{w}_i) + \gamma_i \succcurlyeq R'_j(\bar{w}_j) + \gamma_j \text{ for } i, j = 1, \dots, n; i \neq j.$$

Proof. See Appendix 1.

It follows from Proposition 2 that in the non-trivial equilibrium, the teams winning percentages are determined by their composite strength—the marginal revenue of the winning percentage (R'_i) and the weight parameter (γ_i)—. Following Quirk and Fort (1992, p. 272), we define the marginal revenue of a win for team i as the market size or drawing potential for the team. In line with most of the existing literature, if $R'_i > R'_j$ ($i \neq j$) for any given win percentage, we will refer to team i as the *large-market* (or *strong-drawing*) team and team j as the *small-market* (or *weak-drawing*) team.⁸

First, we consider a special case in which all teams are pure profit-maximizers. Thus, the following corollary follows from Proposition 2.

Corollary 1. Suppose all teams are pure profit-maximizers and satisfy Assumption 1. Then, at the non-trivial Nash equilibrium we have

$$\bar{w}_i [\bar{\ell}_i] \succcurlyeq \bar{w}_j [\bar{\ell}_j] \text{ if and only if } R'_i(\bar{w}_i) \succcurlyeq R'_j(\bar{w}_j) \text{ for } i, j = 1, \dots, n; i \neq j.$$

This corollary implies that if all teams are assumed to be profit-maximizers, the large-market team hires more talent than the small one in the non-trivial equilibrium. Thus, the large-market team will always dominate competition in a league with (pure) profit-maximizing teams. This agrees with the result of Szymanski and Késenne (2004, p.169) for a two-team model. Késenne (2005, p. 103) observed that this result does not

⁸ Burger and Walters (2003) and Krautmann (2009) empirically found that the marginal revenue of the win of a large-market team is larger than that of a small one in Major League Baseball.

necessarily hold in an n -team model. However, it follows from Corollary 1 that Szymanski and Késenne’s results still hold in the general n -team setting.

Second, in view of Proposition 2, it is interesting to note that weak-drawing teams that are more win-oriented can dominate strong-drawing teams that are more profit-oriented, as the following examples demonstrate.

Example 2. Suppose $R_i = m_i w_i$ with $m_i > 0$. Using Proposition 2, it is easily seen that $\bar{w}_i \lesseqgtr \bar{w}_j$ iff $m_i + \gamma_i \lesseqgtr m_j + \gamma_j$. Thus, if $m_i > m_j, \gamma_j > \gamma_i$, and $m_i + \gamma_i < m_j + \gamma_j$ ($i \neq j$), then the small-market team j dominates the large-market team i .

Example 3. Let $R_i = m_i w_i - \frac{b}{2} w_i^2$ with $m_i > 0$ and $b > 0$. The parameter b characterizes the effect of competitive balance on team revenues. Then, in view of Proposition 2, it can be easily demonstrated that in the non-trivial Nash equilibrium $\bar{w}_i \lesseqgtr \bar{w}_j$ iff $m_i - b\bar{w}_i + \gamma_i \lesseqgtr m_j - b\bar{w}_j + \gamma_j$; clearly, this is equivalent to $\bar{w}_i \lesseqgtr \bar{w}_j$ iff $m_i + \gamma_i \lesseqgtr m_j + \gamma_j$. Therefore, the result is same as given in Example 2 above.

Késenne (2004) called this phenomenon the “good” competitive imbalance because sports will be much more attractive, at least for the neutral spectator, when a small-market team succeeds in beating large-market teams. However, Examples 2 and 3 also suggest that if the win preference of large-market teams is larger than or equal to that of the small-market teams, a good imbalance will not occur in a professional sports league. This follows in general from the observation of Proposition 2.

Corollary 2. Suppose Assumption 1 holds for all teams. Then, if the win preference of the large-market team is larger than or equal to that of the small-market team, the large-market team has a higher winning percentage than the small-market team in the non-trivial Nash equilibrium.

Proof. See Appendix 2.

Késenne (2004) called this scenario the “bad” competitive imbalance because a few large-market teams with strong drawing potential dominate the competition year after year. Competitive-balance rules, such as revenue sharing and salary caps, usually attempt to prevent the bad type of imbalance. Although Késenne (2007, pp. 54-55) and Dietl *et al.* (2011) demonstrated Corollary 2, these studies are restricted to two-team models. Therefore, the results of Késenne and Dietl *et al* can be extended to a more general n -team model by Corollary 2.

4. Conclusions

This study has proven that under general conditions, a unique non-trivial Nash equilibrium exists in a contest model of an n -team sports league in which teams maximize a linear combination of profits and wins. Further, we have demonstrated that if the win preference of the large-market team is larger than (or equal to) the small-market team, then the former will dominate the latter in the non-trivial equilibrium. Over the past few years, the Nash equilibrium concept has been used in the analysis of professional team sports. However, many papers restricted attention to two-team models. This study applies the share function approach to a general n -team professional sports model, an approach that avoids the dimensionality problem associated with the best response function approach. In addition, this approach is to be able to analyze sporting contests involving many heterogeneous teams. We believe that the present study may serve as a basis for further theoretical research on professional team sports.

Further research may take the following directions. First, our model has assumed that teams are price takers: the per-unit price of talent is treated parametrically by the teams. However, it might be reasonable to assume that teams have buyer power in the market for talent (Madden, 2011). Hence, we would like to extend our analysis in the presence of labor markets with oligopsony power. Second, a particularly great deal of

attention has been focused on revenue sharing's effects on competitive balance in sports economics literature. However, when the number of teams exceeds by two, revenue sharing's effects on the competitive balance are not clearly described. Késenne (2005) showed that if sports leagues with profit-maximizing teams introduce pool revenue sharing, then the team with the higher number of playing talents before sharing will reduce its demand for talent less than the team with the lower number of playing talents in an n -team model. Moreover, we have demonstrated that if all teams are assumed to be profit-maximizers, the large-market team hires more talent than the small one in the non-trivial equilibrium (Corollary 1). Based on these results, we can conjecture that the result of Szymanski and Késenne (2004) for revenue sharing will carry over in the general n -team setting: revenue sharing leads to less competitive balance. The rigorous proof will be our future research.

Appendix 1

Proof of Proposition 2

Take two teams i and j ($i \neq j$). Consider now an interior equilibrium, then from equation (17),

$$\frac{n}{2}(R'_i + \gamma_i)(1 - \bar{s}_i) - c\bar{T} = 0$$

and

$$\frac{n}{2}(R'_j + \gamma_j)(1 - \bar{s}_j) - c\bar{T} = 0$$

respectively. Dividing the first equation by the second and rearranging the terms, we get

$$\frac{1-\bar{s}_i}{1-\bar{s}_j} = \frac{R'_i + \gamma_j}{R'_i + \gamma_i}. \tag{A1}$$

From (A1), we can assert that $\bar{s}_i \lesseqgtr \bar{s}_j$ if and only if

$$R'_i + \gamma_i \lesseqgtr R'_j + \gamma_j.$$

The proof is completed by observing that $\bar{w}_i = \frac{n}{2}\bar{s}_i$ in context to (3).

Appendix 2

Proof of Corollary 2.

Suppose that if the win preference of a large-market team i is larger than or equal to that of a small-market team j , then $\bar{w}_i \leq \bar{w}_j$ in the non-trivial equilibrium. Then, it must be true that $R'_i(\bar{w}_i) + \gamma_i \leq R'_j(\bar{w}_j) + \gamma_j$ in light of Proposition 2. However, if $\bar{w}_i \leq \bar{w}_j$, we know that $R'_i(\bar{w}_i) + \gamma_i$ is greater than $R'_j(\bar{w}_j) + \gamma_j$, because the marginal revenue curve for the large-market team, team i , lies above the marginal revenue curve for the small-market team, team j , for any given win percentage. Then $\bar{w}_i > \bar{w}_j$ by Proposition 2. This is a contradiction, since we assumed the winning percentage of team j is larger than or equal to that of team i in the non-trivial equilibrium. Therefore, if the win preference of a large-market team i is larger than or equal to that of a small-market team j , then $\bar{w}_i > \bar{w}_j$ in the non-trivial equilibrium.

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